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THE SPEED OF A PLANE COMPRESSION WAVE IN A SOIL

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Introduction.

The purpose of this note is to predict, with help of a simple model, the propagation speed of a plane wave in a soil. Only compression waves are considered and unloading phenomena are neglected. The proposed model idealizes the medium as a mixture of two components, the mixture consisting of solid particles and a gas or a liquid. The composition of the mixture is given in terms of the porosity. It is well known that the porosity of a soil can vary within wide limits. This fact is due to the different structural arrangements soil sediments take on. In the following it is assumed that the porosity changes are entirely due to structural changes of the solid lattice. The speed of the wave is computed by two different methods and shown to agree with a semi-empirical relation first proposed by Wood (Ref 1).

The analysis shows that the propagation speed varies with the porosity of the mixture and can attain a minimum which may lie considerably below the propagation speed of either component. Experimental verification of these conclusions would be desirable.

Analysis.

In order to analyse the motion of a particle when the medium is traversed by a plane compression wave it is expedient to utilize Lagrangian coordinates. The displacement x of a particle from its original undisturbed position a is expressed by $x = x(a,t)$, where t is the time. The strain ϵ is given by

$$\epsilon \equiv \frac{\frac{\partial x}{\partial a} da - da}{da} = \frac{\partial x}{\partial a} - 1 \qquad 1.)$$

When the medium is deformed the conservation of mass principle yields

$$\rho_o da = \rho dx \qquad 2.)$$

or

$$\frac{\rho_o}{\rho} = 1 + \epsilon \qquad 3.)$$

where ρ is the density of the deformed medium and ρ_o is its original density.

The dynamical equation is derived with help of Newton's 2nd law of motion. The velocity of a particle with original abscissa a is given by $u = \frac{\partial x}{\partial t}$. Then the equation of motion of the particle in a stressed state reads

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial x} \qquad 4.)$$

where σ denotes the stress. (If the equation is written for the unstressed state ρ is to be replaced by ρ_o and x by a). It is assumed that there exists a stress-strain

relationship of the medium $\sigma = \sigma(\epsilon, \eta)$, where η characterizes the composition of the mixture in terms of the masses of the components and is defined below. Under the assumption that the composition of the medium remains constant during the motion of the particle one obtains from eqs. 1, 3 and 4

$$\frac{\partial u}{\partial t} = \left\{\left(\frac{\partial \sigma}{\partial \epsilon}\right)_\eta \frac{1}{\rho_o}\right\} \frac{\partial \epsilon}{\partial a} \qquad 5.)$$

Since $\frac{\partial \epsilon}{\partial a} = \frac{\partial^2 x}{\partial a^2}$ and $\frac{\partial u}{\partial t} = \frac{\partial^2 x}{\partial t^2}$ there results the wave equation

$$\frac{\partial^2 x}{\partial t^2} = \left\{\left(\frac{\partial \sigma}{\partial \epsilon}\right)_\eta \frac{1}{\rho_o}\right\} \frac{\partial^2 x}{\partial a^2} \qquad 6.)$$

Equation 5 reveals that $\left(\frac{\partial \sigma}{\partial \epsilon}\right)_\eta \cdot \frac{1}{\rho_o} = \left(\frac{da}{dt}\right)^2$. Hence this quantity represents the square of the speed with which a disturbance shifts from particle to particle when they are in the unstrained state. By definition, the sound speed $\bar{c}$ is the speed with which a disturbance travels relative to the strained particles. Hence it follows from eq. 2 that.

$$\rho_o \frac{da}{dt} = \rho \bar{c} \qquad \text{and}$$

$$\bar{c} = \sqrt{\frac{\rho_o}{\rho^2}\left(\frac{\partial \sigma}{\partial \epsilon}\right)_\eta}$$

If ϵ is replaced by ρ with help of eq. 3 and the stress σ by the pressure $-p$ there results

$$\bar{c} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_\eta} \qquad 7.)$$

In the first derivation the sound speed $\bar{c}$ is evaluated by the following argument. Consider a volume of the mixture and let V_1, m_1 and V_2, m_2 be the partial volumes and masses of the components gas or liquid and the solid respectively. The composition of the medium is characterized by the parameter.

$$\eta = \frac{m_1}{m_2} \tag{8.)}$$

It is assumed that the density of the mixture is given by

$$\rho = \frac{m_1+m_2}{V_1+V_2} = \frac{\eta + 1}{\frac{\eta}{\rho_1} + \frac{1}{\rho_2}} \tag{9.)}$$

where ρ_1 and ρ_2 are the densities of the gas or liquid and the solid component respectively. This ideal mixture law requires that the two components should not interact. This condition is probably violated to some extent in clay like soils. Furthermore the assumption is made that there exist a set of "equations of state" relating density with pressure for the two "pure" components as well as for the mixture (the latter has already been assumed in the derivation of eq. 6). On differentiating eq. 9 with respect to p there results

$$\left(\frac{\partial p}{\partial \rho}\right)_\eta = \frac{(\eta\rho_2 + \rho_1)^2}{[1+\eta][\eta\rho_2^2 \frac{1}{\left(\frac{dp}{d\rho_1}\right)} + \rho_1^2 \frac{1}{\left(\frac{dp}{d\rho_2}\right)}]}$$

But $\left(\frac{\partial p}{\partial \rho}\right)_\eta = \bar{c}^2$ where as $\left(\frac{dp}{d\rho_1}\right)$ and $\left(\frac{dp}{d\rho_2}\right)$ are

by definition the sound speeds in the two "pure" components i.e. c_1 and c_2 respectively. Hence

$$\frac{\bar{c}}{c_2} = \left(\eta \frac{\rho_2}{\rho_1} + 1\right) \sqrt{\frac{1}{[\eta+1][\eta\left(\frac{\rho_2 c_2}{\rho_1 c_1}\right)^2+1]}} \qquad 10.)$$

which yields the sound speed in the mixture as a function of the composition parameter η.

A more conventional measure of the composition is the volume fraction of the first component or the porosity ϵ defined by

$$\epsilon \equiv \frac{V_1}{V_1+V_2} = \frac{\eta\frac{\rho_2}{\rho_1}}{\eta\frac{\rho_2}{\rho_1}+1} \qquad 11.)$$

A minor reduction of eqs. 10 and 11 leads to

$$\frac{\bar{c}}{c_2} = \sqrt{\frac{1}{(\epsilon a+1)(\epsilon b+1)}} \qquad 12.)$$

where

$$a = \frac{\rho_1}{\rho_2} - 1 \qquad \text{and} \qquad b = \frac{\rho_2 c_2^2}{\rho_1 c_1^2} - 1$$

Although the state of the mixture corresponding to $\epsilon = 0$ i.e. the "pure" solid matrix cannot be physically realized it is observed from eq. 12 that $\bar{c} = c_2$ as required. Similarly for $\epsilon = 1$ i.e. the pure gas or liquid, $\bar{c} = c_1$. If either a maximum or a minimum of eq. 12 exists, for $0 < \epsilon < 1$, it is given by

$$\epsilon = -\frac{a+b}{2ab} \qquad 13.)$$

In the following a second derivation of the speed of a plane compression wave in a two phase, two component system is given. This is done with help of the conservation of mass and momentum principles.

Consider a slice of the medium of length dx and of unit cross sectional area which is traversed by a plane compression wave travelling at speed $\bar{c}$ (Fig. 1). It is again assumed that the composition of the mixture remains unchanged on passage of the wave. Let the particle velocities and pressures at station 1 and 2 be respectively $u^{(1)}, p^{(1)}$ and $u^{(2)}=0$, $p^{(2)}$ and $u^{(1)} \ll \bar{c}$. Then the masses $(1-\epsilon)\rho_2\,dx$ of the solid and $\epsilon\rho_1 dx$ of the gas or liquid are accelerated from rest to the velocity $u^{(1)}$ when the wave traverses the medium. The resulting change in momentum is supplied by the pressure difference $(p^{(1)} - p^{(2)})$. Hence the conservation of momentum yields

$$(p^{(1)}-p^{(2)})dt = \left\{\epsilon\rho_1 + (1-\epsilon)\rho_2\right\}\left\{u^{(1)}\right\} \quad dx$$

or since $dx = \bar{c}\,dt$

$$\frac{1}{\bar{c}\left\{\epsilon\rho_1+(1-\epsilon)\rho_2\right\}} = \frac{u^{(1)}}{p^{(1)}-p^{(2)}} \qquad 14.)$$

The conservation of mass gives the following condition. Since the left face of the slice moves at speed $u^{(1)}$ and the right face is stationary the thickness and hence the volume of dx decreases in time dt by $dV_1 = u^{(1)}dt$. This change in volume is due to the following cause. Both the

air or liquid and the solid portion of the medium are compressed by the resulting pressure rise from $p^{(2)}$ to $p^{(1)}$ when the wave has traversed the volume dx. With the assumption that the volume change of the mixture is the sum of the volume changes of the components the resulting decrease in volume is

$$dV_2 = (p^{(1)} - p^{(2)})dx \left\{ \beta_1 \varepsilon + \beta_2(1-\varepsilon) \right\} \qquad 15.)$$

where β_1 and β_2 are the coefficients of compressibility of the two components. Since conservation of mass requires that $dV_1 = dV_2$ there follows that

$$\bar{c} \left\{ \bar{\beta}_1 \varepsilon + \bar{\beta}_2(1-\varepsilon) \right\} = \frac{\ddot{u}^{(1)}}{p^{(1)} - p^{(2)}} \qquad 16.)$$

On combining eqs. 14 and 16

$$(\bar{c})^2 = \frac{1}{\left\{ \rho_1 \varepsilon + \rho_2(1-\varepsilon) \right\} \left\{ \beta_1 \varepsilon + \beta_2(1-\varepsilon) \right\}}$$

and since by definition $\beta = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right) = \frac{1}{\rho c^2}$, there results again after some minor simplifications eq. 12.

In the past the following derivation has been proposed for the propagation speed of an infinitesimal amplitude disturbance in a mixture (Ref 1,2). For a compressible medium the acoustic speed $\bar{c}'$ can be expressed in terms of the coefficient of compressibility β and the density ρ as follows.

$$\bar{c}' = \sqrt{\frac{1}{\beta \rho}} \qquad 17.)$$

The investigators assume that the density of the mixture ρ can be written in terms of the constituent densities as

$$\rho = \rho_1 \epsilon + \rho_2 (1-\epsilon) \qquad 18.)$$

This is equivalent to eq. 9 . In addition an analogous but assertedly empirical relationship is assumed for the compressibility of the mixture i.e.

$$\beta = \beta_1 \epsilon + \beta_2 (1-\epsilon) \qquad 19.)$$

but no justification for this is given. In view of the second development (i.e. eq. 15) this relationship finds an obvious explanation. It is interesting to note that on combining eqs. 17, 18 and 19 again the same result is obtained as given by eq. 12. However in contrast to the assumption underlying eq. 17, eq. 12 holds quite generally for a disturbance of arbitrary amplitude.

In comparing the three derivations as outlined above it appears that the first development given requires in principle nothing more than the density law of the mixture in terms of the constituent densities and a set of equations of state relating density with pressure for mixture and components. The remaining two derivations require in addition a statement relating the compressibility of the mixture in terms of the constituent compressibilities. For the sake of simplicity energy considerations have been completely neglected in the above discussion.

<u>Numerical estimates.</u>

In order to compute $\bar{c}$ as a function of the porosity from eq. 12, it is best to consider ρ_1, c_1 and ρ_2 as known and to determine the propagation speed c_2 in the pure solid matrix from experimental data. For this one requires the value of $\bar{c}$ at a given ϵ. For a soil consisting primarily of solid and air the parameters are probably $\frac{\rho_1}{\rho_2} \simeq 10^{-3}$ and $\frac{c_2}{c_1} \simeq 1$ (for an acoustic disturbance). For these values eq. 12 exhibits a minimum for $\epsilon \simeq \frac{1}{2}$. This corresponds to a mixture in which solid and air occupy approximately the same volume. The acoustic disturbance travels at this minimum far below the propagation speed of either component. For a soil consisting of solid and water the parameters may take on the values $\frac{\rho_1}{\rho_2} \simeq \frac{5}{2}$. and $\frac{c_2}{c_1} \simeq \frac{1}{5}$ (for an acoustic disturbance). The propagation speed exhibits a minimum for $\epsilon \simeq 1/6$. Due to the paucity of experimental data these estimates are probably quite crude. An experimental verification of eq 12, for a Kaolin-water mixture can be found in Ref. 2.

References.

1.) Wood, A. B. A Textbook of Sound. G. Bell and Sons. London (1930) p. 326-29.

2.) Urik, R. J. A sound velocity method of determining the compressibility of finely divided substances, J. Appl. Phys. 18, (1947) p. 983.

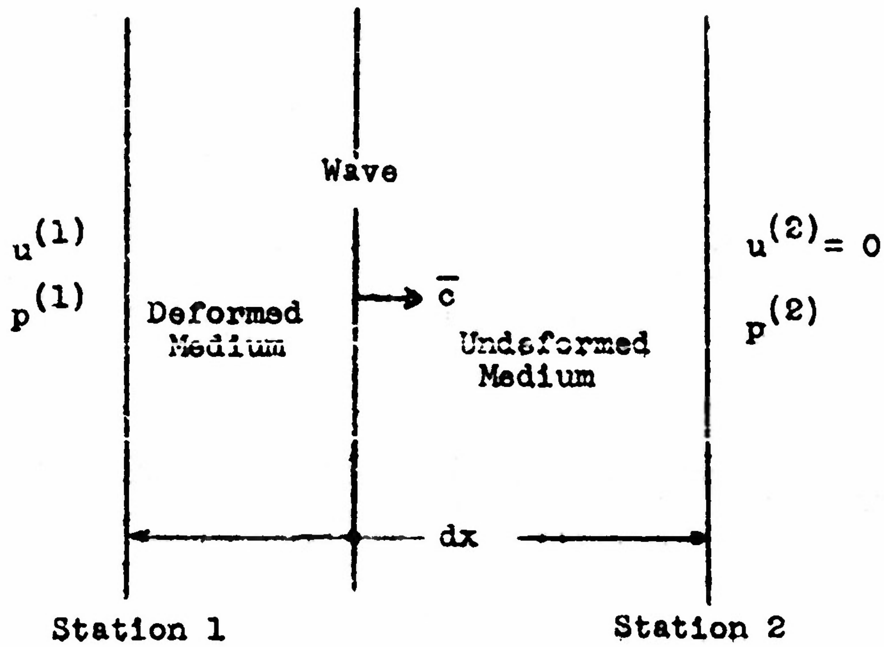
Wave
$u^{(1)}$
$p^{(1)}$
Deformed Medium
$\bar{c}$
Undeformed Medium
$u^{(2)} = 0$
$p^{(2)}$
dx
Station 1
Station 2

Figure 1.